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TWO INEQUALITIES FOR MEDIANS AND ANGLE BISECTORS OF A TRIANGLE

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Abstract. In this paper we prove two inequalities for sum of quotients of angle bisectors and medians of a triangle.

First of all, let us introduce the following notations:
- a, b, c are sides of a triangle;
- A, B, C are vertices of a triangle;
- R, r are radius of circumcircle and of incircle;
- s is semi-perimeter;
- w_a, w_b, w_c are angle bisectors of a triangle and
- m_a, m_b, m_c are medians of a triangle.

Theorem 1. *For any triangle holds:*

$$\sum \frac{w_a}{m_a} > 3 - \sum \left(\frac{a}{b+c} \right)^2 . \tag{1}$$

Proof. $A_1A_2 = \frac{|b-c|a}{2(b+c)}$ where A_1, A_2 are the endpoints of bisector w_a and median m_a on the side BC of triangle ABC. In triangle ABC we have:

$$\frac{w_a}{m_a} \geq 1 - \frac{A_1A_2}{m_a} = 1 - \frac{|b-c|a}{2(b+c)m_a} . \tag{2}$$

The inequality:

$$\frac{|b-c|a}{2(b+c)m_a} < \frac{a^2}{(b+c)^2} \tag{3}$$

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is equivalent to:

$$a^2\left[(b-c)^2+(b+c-a)(b+c+a)\right] > (b-c)^2(b+c)^2\,, \tag{3'}$$

and to:

$$(a+b+c)(a+b-c)(a+c-b)(b+c-a) > 0\,.$$

Inequalities (2) and (3) imply:

$$\frac{w_a}{m_a} > 1-\left(\frac{a}{b+c}\right)^2 \tag{4}$$

From this inequality, (1) follows immediately.

Corollary 1. *In any triangle holds:*

$$\sum\frac{w_a}{m_a} > 1+\frac{16(Rr+r^2)s^2-4R^2r^2}{(s^2+2Rr+r^2)^2}\,. \tag{5}$$

Proof. Bu computing $\sum\left(\frac{a}{b+c}\right)^2$ we get (5), from (1). □

Corollary 2. *For each triangle, hold the inequalities:*

$$\sum\frac{w_a}{m_a} > 1+x\cdot\frac{12x^3+28x^2+31x+16}{(2x^2+3x+2)^2} > 1,\quad \left(x=\frac{r}{R}\right)\,. \tag{6}$$

Proof. *The function given by ratio in (5), as function of S, is monotonic decreasing, and inequality (6) follows from (5) by applying of Gerretsen's inequality:*

$$s^2 \le 4R^2+4Rr+3r^2\,. \tag{7}$$

Remark 1. *Function of x, in (6) is defined in region $0 < x \le \frac{1}{2}$. Minimum 1 of that function is reached in degenerative triangle:*

$$b=c=m_a=w_a,\quad a=w_b=w_c=0,\quad m_b=m_c=\frac{b}{2},$$
$$r=0\quad \textit{and}\quad R=\frac{b}{2}\,.$$

Remark 2. *Because of:*

$$\sum \frac{a}{b+c} > \sum \left(\frac{a}{b+c}\right)^2$$

the inequality

$$\sum \frac{w_a}{m_a} > 3 - \sum \left(\frac{a}{b+c}\right)^2$$

is stronger then the inequality

$$\sum \frac{w_a}{m_a} > 3 - \sum \frac{a}{b+c}$$

given by the same authors in paper [2].

Theorem 2. *In triangle with $a \geq b \geq c$ holds true the inequality:*

$$\sum \frac{w_a}{m_a} > 3 - \frac{2a}{2s-a} \cdot \frac{s^2 - Rr - r^2}{s^2 + 2Rr + r^2}\,. \tag{8}$$

Proof. *If in nonisosceles triangle, we introduce z_a, z_b, z_c by:*

$$\sqrt{1+z_a}|b-c| = 2m_a,\ \sqrt{1+z_b}|c-a| = 2m_b,\ \sqrt{1+z_c}|a-b| = 2m_c \tag{9}$$

and if we put: $z = \min\{z_a, z_b, z_c\}$, then from (2) we have:

$$\frac{w_a}{m_a} > 1 - \frac{1}{\sqrt{1+z_a}} \cdot \frac{a}{b+c} \geq 1 - \frac{1}{\sqrt{1+z}} \cdot \frac{a}{b+c} \tag{10}$$

and, of course:

$$\sum \frac{w_a}{m_a} > 3 - \frac{1}{\sqrt{1+z}} \sum \frac{a}{b+c}$$

Using the equality:

$$\sum \frac{a}{b+c} = 2\frac{s^2 - Rr - r^2}{s^2 + 2Rr + r^2}$$

we get:

$$\sum \frac{w_a}{m_a} > 3 - \frac{2}{\sqrt{1+z}} \cdot \frac{s^2 - Rr - r^2}{s^2 + 2Rr + r^2} \tag{11}$$

from (3) and (9) we have:

$$\frac{a}{2s-a} > \frac{1}{\sqrt{1+z_a}} \cdot \frac{b}{2s-b} > \frac{1}{\sqrt{1+z_b}} \cdot \frac{c}{2s-c} > \frac{1}{\sqrt{1+z_c}}$$

and because of $a > b > c$, also:

$$(12) \qquad \begin{aligned} \frac{a}{2s-a} &= \max\left\{\frac{a}{2s-a}, \frac{b}{2s-b}, \frac{c}{2s-c}\right\} > \\ &> \max\left\{\frac{1}{\sqrt{1+z_a}}, \frac{1}{\sqrt{1+z_b}}, \frac{a}{\sqrt{1+z_c}}\right\} \end{aligned}$$

Inequalities (11) and (12) together imply inequality (8). In the case of equilateral triangle, inequality (8) is true, without introducing of elements z_a, z_b, z_c. In isosceles triangle with two equal sides, as for example with $a > b = c$ we have (10) satisfied with

$$z_a = \infty, \quad \frac{w_a}{m_a} = 1 - \frac{1}{\sqrt{1+z_a}} \cdot \frac{a}{b+c} > 1 - \frac{1}{\sqrt{1+z}} \cdot \frac{a}{b+c}$$

and the rest of the proof is the same as in the case of nonisosceles triangle.

Corollary 3. *Using the inequality (7) in (8) we get:*

$$(13) \qquad \sum \frac{w_a}{m_a} > 3 - \frac{a}{2s-a} \cdot \frac{4R^2 + 3Rr + 2r^2}{2R^2 + 3Rr + 2r^2}.$$

Remark 1. *Since $\frac{a}{2s-a} < 1$, inequality (13) is stronger than inequality*

$$\sum \frac{w_a}{m_a} > \frac{s^2 + 8Rr + 5r^2}{s^2 + 2Rr + r^2}$$

given in the paper [2].

1. References

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